

8.4

几何应用与方向导数

同步练习

8.4.一

1. 曲线 $x = \cos t, y = \sin 2t, z = \cos 3t$ 在 $t = \frac{\pi}{4}$ 处的切线方程_____

解 当 $t = \frac{\pi}{4}$ 时, $x_0 = \frac{\sqrt{2}}{2}, y_0 = 1, z_0 = -\frac{\sqrt{2}}{2},$

$$x'(t_0) = -\sin t_0 = -\frac{\sqrt{2}}{2}$$

$$y'(t_0) = 2\cos 2t_0 = 0,$$

$$z'(t_0) = -3\sin 3t_0 = -\frac{3\sqrt{2}}{2}$$

切线方程为 $\frac{x - \frac{\sqrt{2}}{2}}{1} = \frac{y - 1}{0} = \frac{z + \frac{\sqrt{2}}{2}}{3}$

8.4.一

2. 曲面 $z = x^2 - y^2$ 在点 $(1, 2, -3)$ 处的切平面方程_____

解 $F(x, y, z) = z - x^2 + y^2,$

$$F_x(M_0) = -2x|_{M_0} = -2,$$

$$F_y(M_0) = 2y|_{M_0} = 4, \quad F_z(M_0) = 1$$

法向量 $\vec{n} = \{-2, 4, 1\}$

曲面切平面方程为 $-2(x-1) + 4(y-2) + z + 3 = 0$

$$2x - 4y - z + 3 = 0$$

8.4.一

3. 函数 $f(x, y) = xe^{2y}$ 在点 $P(1, 0)$ 处沿 $l = \{-3, 4\}$ 方向的方向导数 $\frac{\partial f}{\partial l} \Big|_P$.

解 先求向量 $\vec{l}$ 的方向余弦 $\cos \alpha = -\frac{3}{5}, \cos \beta = \frac{4}{5}$

再函数在 P 点的偏导数值

$$f_x \Big|_P = e^{2y} \Big|_P = 1 \quad f_y \Big|_P = 2xe^{2y} \Big|_P = 2$$

$$\frac{\partial f}{\partial l} \Big|_P = \frac{\partial f}{\partial x} \Big|_P \cos \alpha + \frac{\partial f}{\partial y} \Big|_P \cos \beta = -\frac{3}{5} + \frac{8}{5} = 1$$

8.4.一

4. 函数 $f(x, y) = \frac{x^2 + y^2}{2}$ 在点 $P(1, 1)$ 处的梯度 $\text{grad}f(P) = \underline{\hspace{2cm}}$

解 $f_x|_P = x|_P = 1$

$$f_y|_P = y|_P = 1$$

$$\text{grad}f(P) = \{f_x(P), f_y(P)\}$$

$$= \{1, 1\}$$

$$= (1, 1) = \vec{i} + \vec{j}$$

8.4.二

1. 求曲线 $\begin{cases} y = 2x^2 \\ z = x^3 \end{cases}$ 在点(1,2,1)处的切线和法平面方程

解 当 $x = 1$ 时, $y'(x_0) = 4x_0 = 4$,
 $z'(x_0) = 3x_0^2 = 3$

切线方程为 $\frac{x-1}{1} = \frac{y-2}{4} = \frac{z-1}{3}$

法平面方程为

$$x - 1 + 4(y - 2) + 3(z - 1) = 0$$

$$\text{即 } x + 4y + 3z - 12 = 0$$

8.4.二

2. 求曲线 $\begin{cases} x^2 + y^2 + z^2 = 4 \\ x^2 + y^2 = 2x \end{cases}$ 在点 $P(1,1,\sqrt{2})$ 处的切线与法平面方程

解令 $\begin{cases} F = x^2 + y^2 + z^2 - 4 \\ G = x^2 + y^2 - 2x \end{cases}$ 雅可比矩阵

切向量为 $\vec{T} \quad \begin{pmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \end{pmatrix} \Big|_P = \begin{pmatrix} 2x & 2y & 2z \\ 2x-2 & 2y & 0 \end{pmatrix} \Big|_P$

$$= \left\{ \begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}_P, \begin{vmatrix} F_z & F_x \\ G_z & G_x \end{vmatrix}_P, \begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}_P \right\} = \begin{pmatrix} 2 & 2 & 2\sqrt{2} \\ 0 & 2 & 0 \end{pmatrix}$$

$$= \{-4\sqrt{2}, 0, 4\}$$

$$\text{切线方程为 } \frac{x-1}{\sqrt{2}} = \frac{y-1}{0} = \frac{z-\sqrt{2}}{-1}$$

$$\text{法平面方程为 } \sqrt{2}(x-1) - z + \sqrt{2} = 0$$

$$\text{即 } \sqrt{2}x - z = 0$$

8.4.二

3. 在曲线 $x = t, y = -t^2, z = t^3$ 上求一点,使得曲线在该点处的切线平行于平面 $x + 2y + z = 4$,并求切线方程

解 $x'(t_0) = 1, y'(t_0) = -2t_0, z'(t_0) = 3t_0^2$

切向量为 $\vec{T} = \{1, -2t_0, 3t_0^2\}$

由于与平面平行 故 $1 \cdot 1 + (-2t_0) \cdot 2 + 3t_0^2 \cdot 1 = 0$

解得 $t_{01} = 1, t_{02} = \frac{1}{3}$ 切点坐标为 $(1, -1, 1)$ 或 $(\frac{1}{3}, -\frac{1}{9}, \frac{1}{27})$

切线方程为 $\frac{x-1}{1} = \frac{y+1}{-2} = \frac{z-1}{3}$

或 $\frac{x-\frac{1}{3}}{3} = \frac{y+\frac{1}{9}}{-2} = \frac{z-\frac{1}{27}}{1}$

8.4.二

4.在曲面 $e^z + xy = z + 3$ 在点 $P(2,1,0)$ 处的切平面与法线方程

解 先求切平面的法向量

$$\text{设 } F = e^z + xy - z - 3$$

$$F_x \big|_P = y \big|_P = 1 \quad F_y \big|_P = x \big|_P = 2$$

$$F_z \big|_P = e^z - 1 \big|_P = 0 \quad \text{法向量 } \vec{n} = \{1, 2, 0\}$$

$$\text{切平面方程 } x - 2 + 2(y - 1) = 0 \quad \text{即 } x + 2y - 4 = 0$$

$$\text{法线方程 } \frac{x-2}{1} = \frac{y-1}{2} = \frac{z-0}{0}$$

8.4.二

5. 在曲面 $z = x^2 + \frac{y^2}{2}$ 上求一点, 使得曲面在该点处的切平面平行于平面 $2x + 2y - z = 0$, 并求切平面方程

解 先求曲面的法向量 设 $F = x^2 + \frac{y^2}{2} - z$

$$F_x = 2x \quad F_y = y \quad F_z = -1$$

法向量 $\vec{n} = \{2x, y, -1\}$ 于是 $\frac{2x}{2} = \frac{y}{2} = 1$

解得 $x = 1, y = 2, z = x^2 + \frac{y^2}{2} = 3.$

切点为 $(1, 2, 3)$

切平面方程 $2(x - 1) + 2(y - 2) - z + 3 = 0$

$$\text{即 } 2x + 2y - z - 3 = 0$$

8.4.二

6. 设直线 $l: \begin{cases} x+y+b=0 \\ x+ay-z-3=0 \end{cases}$ 在平面 Π 上, 平面 Π 与曲面 $z = x^2 + y^2$ 切于点 $P(1, -2, 5)$, 求 a, b 的值.

解 先曲面的法向量 设 $F = x^2 + y^2 - z$

$$F_x|_P = 2x|_P = 2 \quad F_y|_P = 2y|_P = -4$$

$$F_z|_P = -1|_P = -1 \quad \text{法向量 } \vec{n} = \pm\{2, -4, -1\}$$

平面 Π 的方程: $x + y + b + \lambda(x + ay - z - 3) = 0$

$$\text{于是 } 1 - 2 + b + \lambda(1 - 2a - 5 - 3) = 0$$

$$\frac{2}{1 + \lambda} = -\frac{4}{1 + a\lambda} = \frac{1}{\lambda}$$

解得 $\lambda = 1, a = -5, b = -2.$

8.4.二

7.求曲面 $3x^2 + y^2 + z^2 = 16$ 上点 $P(-1, -2, 3)$ 处的切平面与 xoy 坐标面的夹角.

解 先求曲面的法向量 设 $F = 3x^2 + y^2 + z^2 - 16$

$$F_x|_P = 6x|_P = -6$$

$$\text{法向量}\vec{n} = \pm\{-3, 2, 3\}$$

$$F_y|_P = 2y|_P = -4$$

$$\cos \alpha = \mu \frac{3}{\sqrt{22}}, \quad \cos \beta = \pm \frac{2}{\sqrt{22}},$$

$$F_z|_P = 2z|_P = 6$$

$$\cos \gamma = \pm \frac{3}{\sqrt{22}}$$

由于切平面与 xoy 坐标面的夹角 $\varphi \in [0, \frac{\pi}{2}]$

$$\cos \varphi = |\cos \gamma|$$

$$\varphi = \arccos \frac{3}{\sqrt{22}}$$

8.4.二

8. 求函数 $u = \ln(x + \sqrt{y^2 + z^2})$ 在点 $P(1,0,1)$ 处沿 P 指向点 $B(2,-2,3)$ 方向的方向导数.

解 先求向量 $\overline{PB}$ 的方向余弦

$$\overline{PB} = \{1, -2, 2\} \quad \cos \alpha = \frac{1}{3}, \cos \beta = -\frac{2}{3}, \cos \gamma = \frac{2}{3}$$

$$\text{再函数在 } P \text{ 点的偏导数值 } u_x|_P = \frac{1}{x + \sqrt{y^2 + z^2}} \Big|_P = \frac{1}{2}$$

$$u_y|_P = \frac{1}{x + \sqrt{y^2 + z^2}} \frac{y}{\sqrt{y^2 + z^2}} \Big|_P = 0$$

$$u_z|_P = \frac{1}{x + \sqrt{y^2 + z^2}} \frac{z}{\sqrt{y^2 + z^2}} \Big|_P = \frac{1}{2}$$

$$\frac{\partial f}{\partial l} \Big|_P = \frac{\partial f}{\partial x} \Big|_P \cos \alpha + \frac{\partial f}{\partial y} \Big|_P \cos \beta + \frac{\partial f}{\partial z} \Big|_P \cos \gamma = \frac{1}{6} + 0 + \frac{1}{3} = \frac{1}{2}$$

8.4.二

9. 求函数 $u = xy^2 + z^3 - xyz$ 在点 $P(1,1,1)$ 处沿曲面 $x^2 + 2y^2 + z^2 = 4$ 在 P 处外法向量方向的方向导数.

解 先求曲面在 P 点处外法向量方向余弦

$$F = x^2 + 2y^2 + z^2 - 4 \quad F_x|_P = 2x|_P = 2 \quad F_y|_P = 4x|_P = 4$$

$$F_z|_P = 2z|_P = 2$$

法向量方向余弦

$$\cos \alpha = \frac{1}{\sqrt{6}}, \cos \beta = \frac{2}{\sqrt{6}}, \cos \gamma = \frac{1}{\sqrt{6}}$$

再函数在 P 点的偏导数值

$$u_x|_P = y^2 - yz|_P = 0 \quad u_y|_P = 2xy - xz|_P = 1$$

$$u_z|_P = 3z^2 - xy|_P = 2$$

$$\frac{\partial f}{\partial l}|_P = \frac{\partial f}{\partial x}|_P \cos \alpha + \frac{\partial f}{\partial y}|_P \cos \beta + \frac{\partial f}{\partial z}|_P \cos \gamma$$

$$= 0 + \frac{2}{\sqrt{6}} + \frac{2}{\sqrt{6}} = \frac{4}{\sqrt{6}} = \frac{2}{3}\sqrt{6}$$

8.4.二

10. 求函数 $u = xy^2z$ 在点 $P(1,-1,2)$ 处的梯度.

解 $u_x|_P = y^2z = 2$ $\text{gradu}(P) = \{u_x(P), u_y(P), u_z(P)\}$

$$u_y|_P = 2x^2z = 4$$
$$= \{2, 4, -1\}$$

$$u_z|_P = xy^2 = -1$$
$$= (2, 4, -1) = 2\vec{i} + 4\vec{j} - \vec{k}$$

8.4.三

证明曲面 $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a} (a > 0)$ 上任一点处的切平面在三个坐标轴上的截距之和为常数

解 设曲面上任一点为 (x_0, y_0, z_0) , 则 $\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = \sqrt{a}$

$$\text{令 } F(x, y, z) = \sqrt{x} + \sqrt{y} + \sqrt{z} - \sqrt{a}$$

$$\text{则 } F'_x = \frac{1}{2\sqrt{x}}, F'_y = \frac{1}{2\sqrt{y}}, F'_z = \frac{1}{2\sqrt{z}}$$

$$\Rightarrow \text{法向量为 } \vec{n} = \left\{ \frac{1}{2\sqrt{x_0}}, \frac{1}{2\sqrt{y_0}}, \frac{1}{2\sqrt{z_0}} \right\}$$

$$\Rightarrow \text{切平面方程为 } \frac{1}{2\sqrt{x_0}}(x - x_0) + \frac{1}{2\sqrt{y_0}}(y - y_0) + \frac{1}{2\sqrt{z_0}}(z - z_0) = 0$$

$\Rightarrow$ 切平面在三个坐标轴上的截距分别为

$$a = x_0 + \sqrt{x_0 y_0} + \sqrt{x_0 z_0}$$

$$b = y_0 + \sqrt{x_0 y_0} + \sqrt{y_0 z_0},$$

$$c = z_0 + \sqrt{x_0 z_0} + \sqrt{y_0 z_0}$$

$$\text{故截距和为 } a + b + c = x_0 + y_0 + z_0 + 2\sqrt{x_0 y_0} + 2\sqrt{x_0 z_0} + 2\sqrt{y_0 z_0}$$

$$= (\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0})^2 = a, \text{ 结论成立}$$

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